

Quantum mechanics is based on a relativity principle

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Non-relativistic quantum mechanics is shown to emerge from classical mechanics through the requirement of a relativity principle based on special transformations acting on position and momentum uncertainties. These transformations are related to dilatations of space variables provided the quantum potential is added to the classical Hamiltonian functional. The Schrödinger equation appears to have a nonunitary and nonlinear companion acting in another time variable. Evolution in this time seems related to the state vector reduction

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Unlike special relativity and the Einstein's theory of gravitation, quantum mechanics is not considered to be based on a principle of relativity. The formers require the invariance of the mathematical representation of the laws of Nature under transformations relating so-called observers or space-time frames of reference. The demand that classical mechanics becomes invariant under these transformations entails modifications of the fundamental laws of physics.

Our claim in this Letter is that quantum mechanics is also a relativity theory in the sense that this theory emerges out of classical mechanics from the condition of invariance of the laws of mechanics under a new group of frame transformations. This group acts on the precision of position and momentum measurements of the observer. Observers or frames of references are characterized not only by the position of their space and time origins, direction of their axis and relative velocities but also by the accuracy or resolution of their instruments. Such an approach to quantum mechanics has not been contemplated often in modern physics. The most notorious exception is, to our knowledge, the interesting work of Nottale [1],[2],[3]. The theory reported here, however, differs fundamentally from that developed by Nottale.

This difference is mainly due to the fact that we do not attribute a fractal character to space-time in contrast with this researcher. In Nottale's work, this fundamental fractality is inferred from the discovery by Abbott and Wise that the trajectory of a quantum free particle has fractal dimension 2 [4]. Nottale relates this characteristics to a fundamental fractal property of space-time. Due to this fractality, he is led to consider scale transformations. These transformations are, however, very different from ours. Moreover, the meaning of a fractal trajectory is somewhat vague in the sense that many aspects of quantum mechanics point to the vacuity of the concept of trajectory. In contrast, our work does not invoke such concept and remains in the usual Copenhagen interpretation frame. There is, however, a possible articulation with Nottale's work in the sense that starting with a Galilean space-time and requiring only invariance of physics under a well-defined relativity group acting on

measurements precision, we find a supplementary time dimension. Hence, "trajectories" are defined by two temporal real parameters and can be viewed as 2-dimensional objects. But this affects the topological dimension and not the fractal dimension. This is, perhaps, the limit between the two approaches.

Due to the concision inherent to a Letter the demonstrations of many results reported here are only outlined, though, all of them can be retraced by the reader. A more complete article will be submitted elsewhere containing also generalization to a particle in an exterior potential and to systems of N particles interacting via a binary potential.

Let us consider a non-relativistic free particle described by the Schrödinger equation. The transformations on the uncertainties we are proposing are the following

$$\Delta x'^2 = e^{-\alpha} \Delta x^2 \quad (1)$$

$$\Delta p'^2 = e^{-\alpha} \Delta p^2 + \frac{\hbar^2}{4} (e^{\alpha} - e^{-\alpha}) \frac{1}{\Delta x^2} \quad (2)$$

where Δx^2 is the Fisher dispersion of the position probability distribution (p.d.) $\rho(x)$. It is given by

$$\Delta x^2 = \frac{1}{2 \int |\nabla \rho^{1/2}(x)|^2 d^3x} \quad (3)$$

where the denominator is the Fisher information associated to the p.d. ρ [5].

Its relation with the position mean square deviation σ_x^2 is given by the Cramér-Rao inequality

$$\sigma_x^2 \geq \Delta x^2 \quad (4)$$

a classical result in theoretical statistics obtained from the Schwartz inequality [6].

The quantity Δp^2 is the mean square deviation of momentum calculated with the standard quantum algorithm for the expectation of any function of the momentum operator. The group property of the above transformations is easy to establish. From here on, we consider it as the relativity group relating the various observers and under which the laws of physics should be covariant.

Multiplying equation (1) by equation (2) one gets

$$\Delta x'^2 \Delta p'^2 = e^{-2\alpha} \Delta x^2 \Delta p^2 + \frac{\hbar^2}{4} (1 - e^{-2\alpha}) \quad (5)$$

The parameter α is any real number. The one-parameter continuous group structure of the set of these transformations is easy to prove. Furthermore, when $\alpha \rightarrow +\infty$, $\Delta x'^2 \Delta p'^2 \rightarrow \frac{\hbar^2}{4}$. If $\Delta x^2 \Delta p^2$ is already equal to $\frac{\hbar^2}{4}$ then the product $\Delta x'^2 \Delta p'^2$ keeps the value $\frac{\hbar^2}{4}$ for any value of α . For $\alpha \rightarrow -\infty$, $\Delta x'^2 \Delta p'^2 \rightarrow +\infty$ for any value of $\Delta x^2 \Delta p^2 \geq \frac{\hbar^2}{4}$. Of course, since the Cramér-Rao inequality (4) guarantees that $\sigma_x^2 \Delta p^2 \geq \Delta x^2 \Delta p^2$, all these asymptotic results are lower boundaries for the values of $\sigma_x^2 \Delta p^2$ and its transformations.

These remarkable properties bear some similarities with the Lorentz transformations. In the same way the velocity of light constitutes an upper value for the velocities of material bodies, the constant $\frac{\hbar^2}{4}$ represents a lower limit value for the product of uncertainties $\Delta x^2 \Delta p^2$.

The choice of the above transformations (1), (2) as relativity group for the laws of physics imposes a radical modification of the laws of dynamics that corresponds to the passage from classical to quantum mechanics as we prove now.

Let us consider the classical mechanical description of a free non-relativistic particle of mass m . We describe it in a field canonical framework [7] by introducing at the initial time the p.d. $\rho(x)$ of an ensemble of identical non-interacting particles. This function together with the classical action of the particle, $s(x)$, are the basic field variables of the formalism. The time evolution of any functional of type

$$\mathcal{A} = \int d^3x F(x, \rho, \nabla \rho, \nabla \nabla \rho, \dots, s, \nabla s, \nabla \nabla s, \dots) \quad (6)$$

of the two variables ρ and s that is at least once functionally differentiable in terms of ρ and s is given by

$$\partial_t \mathcal{A} = \{\mathcal{A}, \mathcal{H}_{cl}\} \quad (7)$$

where

$$\mathcal{H}_{cl} = \int d^3x \frac{\rho |\nabla s|^2}{2m} \quad (8)$$

is the classical Hamiltonian functional and

$$\{\mathcal{A}, \mathcal{B}\} = \int d^3x \left[\frac{\delta \mathcal{A}}{\delta \rho(x)} \frac{\delta \mathcal{B}}{\delta s(x)} - \frac{\delta \mathcal{B}}{\delta \rho(x)} \frac{\delta \mathcal{A}}{\delta s(x)} \right] \quad (9)$$

where $\frac{\delta}{\delta \rho(x)}$ and $\frac{\delta}{\delta s(x)}$ are functional derivatives. The above functional Poisson bracket endows the set of functionals of type (6) with an infinite Lie algebra structure \mathbb{G} .

Any functional of \mathbb{G} , and \mathcal{H}_{cl} is one of them, generates a one-parameter continuous group of transformations.

The time transformations are generated by \mathcal{H}_{cl} . Equation (7) when applied to $\rho(x)$ and $s(x)$ respectively, yields the continuity equation and the Hamilton-Jacobi equation

$$\partial_t \rho = -\nabla \cdot \left(\rho \frac{\nabla s}{m} \right) \quad (10)$$

$$\partial_t s = -\frac{|\nabla s|^2}{2m} \quad (11)$$

where the gradient ∇s is the momentum of the particle.

Now let us consider the group of space dilatations and its action on ρ and s

$$\rho'(x) = e^{\frac{3\alpha}{2}} \rho(e^{\frac{\alpha}{2}} x), s'(x) = e^{-\alpha} s(e^{\frac{\alpha}{2}} x) \quad (12)$$

where α is any real number. Note that these transformations preserve the normalization of the p.d. $\rho(x)$ [5]. Clearly, they also keep the dynamical equations (10) and (11) invariant.

Let us assume that the average momentum of the particle is vanishing. This corresponds to a particular choice of the frame of reference but, by no means, reduces the generality of our results. In this frame, the quadratic mean deviation of the momentum is given by

$$\Delta p_{cl}^2 = \int d^3x \rho |\nabla s|^2 = 2m \mathcal{H}_{cl} \quad (13)$$

and under transformations (12) becomes

$$\Delta p_{cl}'^2 = e^{-\alpha} \Delta p_{cl}^2 \quad (14)$$

Also, the Fisher dispersion of $\rho(x)$, Δx^2 , defined in equation (3) transforms as

$$\Delta x'^2 = e^{-\alpha} \Delta x^2 \quad (15)$$

Not surprisingly, it appears from equation (14) that the classical momentum uncertainty does not transform like prescribed by equation (2) above. In view of the physical dimensions of Δp_{cl}^2 as defined in equation (13), transformation (14) is expected under dilatations affecting the position coordinates. It corresponds to the first term in the right hand side of equation (2).

Now, let us modify definition (13) of Δp_{cl}^2 by adding a new term proportional to the Fisher information, $\frac{\hbar^2}{2} F$. This addition along with the use of equation (3) yields the following quantity

$$\Delta p_q^2 = \int d^3x \rho(x) |\nabla s(x)|^2 + \hbar^2 \int d^3x |\nabla \rho(x)^{1/2}|^2 \quad (16)$$

We now prove that the above supplementary term restores the relativity transformation law (2). Let us apply the space dilatation (12) to this functional. This leads to

$$\Delta p_q'^2 = e^{-\alpha} \int d^3x \rho(x) |\nabla s(x)|^2 + e^{\alpha} \hbar^2 \int d^3x |\nabla \rho(x)^{1/2}|^2 \quad (17)$$

Adding and subtracting an appropriate term, $e^{-\alpha} \hbar^2 \int d^3x |\nabla \rho(x)^{1/2}|^2$, to the right hand side of equation (17) allows expressing this equation in terms of Δp_q^2 as follows

$$\Delta p'_q{}^2 = e^{-\alpha} \Delta p_q^2 + \frac{\hbar^2}{4} (e^\alpha - e^{-\alpha}) \frac{1}{\Delta x^2} \quad (18)$$

where we have used again equation (3). This equation is identical to the transformation law (2). We may, thus, identify Δp_q with Δp , i.e. the quantal momentum uncertainty. Since the Hamiltonian functional for a free particle is its average kinetic energy, we have in this frame

$$\mathcal{H}_q = \frac{\Delta p_q^2}{2m} \quad (19)$$

or

$$\mathcal{H}_q = \int d^3x \frac{\rho(x) |\nabla s(x)|^2}{2m} + \frac{\hbar^2}{2m} \int d^3x |\nabla \rho(x)^{1/2}|^2 \quad (20)$$

This is precisely the expected expression of the quantum average of the energy for a free particle. Clearly, the apparition of the Planck constant in this derivation is quite artificial. The constant multiplying the Fisher information in the added term in (16) is just arbitrary. The Planck value of that constant has been postulated in order to retrieve quantum mechanics.

The functional \mathcal{H}_q generates the quantum time evolution of any functional \mathcal{A} of the algebra \mathbb{G} via equation (7) where \mathcal{H}_{cl} is to be replaced by \mathcal{H}_q . In particular it gives the Schrödinger equation when \mathcal{A} is just the wave function $\rho^{1/2} e^{is/\hbar}$. We leave this demonstration to the reader. One of the intermediate results is the apparition of the quantum potential [8] in the Hamilton-Jacobi equation (11)

$$\partial_t s = -\frac{|\nabla s|^2}{2m} + \frac{\hbar^2}{2m} \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} \quad (21)$$

while the continuity equation for ρ (10) is preserved.

Let us summarize. We have derived the quantum evolution for a free particle from the requirement that the quadratic uncertainties on position and momentum should satisfy the relativity transformations laws (1) and (2). This result is easily generalized to a particle in an exterior potential or to N particles interacting via a binary potential. The form in which we obtain quantum mechanics is that of canonical field theory which has been introduced and studied from different points of view by various authors [9], [10], [11], [12], [13]. None of these authors, however, derives quantum mechanics from a relativity principle as we do here. They assume the existence of the quantum formalism based on Hilbert space and the algebra of observable operators acting on it, and show that this framework can be derived from a more

general symplectic or canonical field theory and/or from a variational principle.

One more important question has now to be investigated: That of the non-invariance of the Schrödinger equation under the transformation (1), (2). It is a well-known fact that this equation is not invariant under the conformal group and our transformations are, indeed, dilatations of position coordinates. Let us first remark that the generator of transformations (1) and (2) in the algebra \mathbb{G} is the functional

$$\mathcal{S} = \int d^3x \rho(x) s(x) \quad (22)$$

This can readily be verified by exponentiating the infinitesimal transformation

$$\Delta p'_q{}^2 = \Delta p_q^2 + \delta\alpha \{ \Delta p_q^2, \mathcal{S} \} \quad (23)$$

where the bracket is still the one defined in equation (9). So doing, one gets the same expression as equation (17) or (2). Let us now define the following new functional

$$\mathcal{K}_q \equiv \{ \mathcal{S}, \mathcal{H}_q \} = \int d^3x \frac{\rho |\nabla s|^2}{2m} - \frac{\hbar^2}{2m} \int d^3x |\nabla \rho^{1/2}|^2 \quad (24)$$

and let us apply the group generated by \mathcal{S} on both \mathcal{H}_q and \mathcal{K}_q . An easy calculation yields

$$\mathcal{H}'_q = \cosh\alpha \mathcal{H}_q - \sinh\alpha \mathcal{K}_q, \mathcal{K}'_q = -\sinh\alpha \mathcal{H}_q + \cosh\alpha \mathcal{K}_q \quad (25)$$

This is due to the fact that $\{ \mathcal{S}, \mathcal{K}_q \}$ is equal to \mathcal{H}_q . These transformations are isomorphic to 2-D Lorentz transformations.

Since \mathcal{K}_q only differs from \mathcal{H}_q by the sign of the quantum potential, the group it generates is parametrized by a new time parameter, τ . Any functional \mathcal{A} of \mathbb{G} can be considered as a function of both t and τ and its evolution in both times is given by

$$\partial_t \mathcal{A} = \{ \mathcal{A}, \mathcal{H}_q \}, \partial_\tau \mathcal{A} = \{ \mathcal{A}, \mathcal{K}_q \} \quad (26)$$

Note also that both generators tend to \mathcal{H}_{cl} for $\hbar \rightarrow 0$, i.e. both times variables become identical in the classical limit. For finite value of \hbar the transformations (25) induce Lorentz-like transformations in the plane (t, τ)

$$t' = \cosh\alpha t + \sinh\alpha \tau, \tau' = \sinh\alpha t + \cosh\alpha \tau \quad (27)$$

Also, the remarkable property that $\mathcal{H}_q + i\mathcal{K}_q$ is a holomorphic function of $t + i\tau$ is easily shown.

The transformations generated by \mathcal{S} mix the two time evolutions, hence, only system (26) is covariant but not the individual equations constituting it. Let us now consider the case where \mathcal{A} is the wave function ψ given by $\rho^{1/2} e^{is/\hbar}$. As stated above its evolution equation in variable t is linear and is Schrödinger's equation. In time τ , however, the equation is nonlinear

$$i\hbar \partial_\tau \psi = -\frac{\hbar^2}{2m} \nabla^2 \psi + \frac{\hbar^2}{m} \psi \frac{\nabla^2 |\psi|}{|\psi|} \quad (28)$$

The nonlinear Schrödinger equation obtained here is not a newcomer in physics. It has been envisaged, though in the time t variable and in different contexts, by several authors [14], [15], [16]. It belongs to the class of Weinberg's nonlinear Schrödinger equations [17]. This equation admits a nonlinear superposition principle [18]. It has been studied, always in the usual time variable, as a member of the general class of nonlinear Schrödinger equations obtained under the so-called nonlinear gauge transformations introduced by Doebner and Goldin [19]. The evolution generated by this equation in our new time dimension is nonunitary as \mathcal{K}_q can not be reduced to the quantum average of a Hermitian operator. One easily shows also that together with the functionals generating translations, rotations and Galilean boosts, \mathcal{K}_q constitutes a field canonical representation of the Galilei algebra. A potentially important property is that equation (29) implies the continuity equation for the p.d. ρ .

Moreover, the functional \mathcal{S} is a Lyapunov-like function for this equation as

$$\partial_\tau \mathcal{S} = \{\mathcal{S}, \mathcal{K}_q\} = \mathcal{H}_q \geq 0 \quad (29)$$

This property is related to the fact that while $\partial_t \Delta x^2 \geq 0$ and $\partial_t \Delta p^2 = 0$, the product $(\partial_\tau \Delta x^2)(\partial_\tau \Delta p^2)$ is always negative. This is reminiscent of the process of state vector reduction in position measurement in which $\Delta x^2 \rightarrow 0$ while $\Delta p^2 \rightarrow +\infty$, or conversely if one is measuring momentum. Would this evolution correspond to the nonunitary process that authors like R. Penrose [20] are invoking for the description of the collapse of the wave function? The difference with these approaches lies, at least, in the fact that they always consider the reduction process in the usual time.

Several directions of generalization of our theory can be envisaged. One would be abandoning global invariance with respect to transformations (1) and (2) and requiring only local invariance. This could lead to the discovery of a new gauge field. Another orientation would be the extension of the above approach to the case of Klein-Gordon and Dirac equations, and more generally to quantum field theory with, perhaps, important conse-

quences at the level of the general unified theory including gravitation.

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